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# Timestepping schemes for the 3d Navier–Stokes equations

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## Abstract

It is well known that the solutions of the 3d Navier–Stokes equations remain bounded for all time if the initial data and the forcing are sufficiently small relative to the viscosity, and for a finite time given any bounded initial data. In this article, we consider two temporal discretisations (semi-implicit and fully implicit) of the 3d Navier–Stokes equations in a periodic domain and prove that their solutions remain bounded in  $H^1$  subject to essentially the same respective smallness conditions (on initial data and forcing or on the time of existence) as the continuous system and to suitable timestep restrictions.

*Keywords:* 3d Navier–Stokes, small solutions, short time, temporal discretisation, Euler schemes

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## 1. Introduction

Much work has been done on the stability and convergence of various timestepping schemes for the Navier–Stokes equations in two space dimensions (2d NSE). The long time stability of Euler schemes for the 2d NSE has been treated in, e.g.,  
5 [2, 6, 4, 8], and more recently extended to higher-order schemes in [9, 3]. Once the numerical solutions are shown to be bounded in suitable space, either on a

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limited interval of time or for all time, convergence can usually be established using standard techniques (cf., e.g., [5]).

In three dimensions, the existence of a solution bounded in  $L^\infty(\mathbb{R}_+, H^1)$  is not known, we only know a globally bounded solution with small data and of a local bounded solution with arbitrary data; for more background on the NSE, see e.g. [1, 7]. Hence, the extension of the numerical stability results from 2d or 3d is not straightforward. We conduct it here in the two cases for which the existence of strong solution is known, namely, as we said, a globally bounded solution with small data or a locally bounded solution with arbitrary data.

In this article we consider temporal discretisations of the 3d NSE using the semi-implicit (2.1) and fully implicit (3.1) Euler schemes, and their boundedness in  $H^1$  for interval of times corresponding to the existence of solutions. As in the earlier works cited above, we do not consider spatial discretisations, giving the advantage that our results will be free of Courant–Friedrichs–Lewy-type constraints, although some smallness of the timestep may be required.

We consider the Navier–Stokes equations in  $\Omega = (0, 2\pi)^3$  with periodic boundary conditions,

$$\begin{aligned}\partial_t u + (u \cdot \nabla)u + \nabla p &= \nu \Delta u + f, \\ \nabla \cdot u &= 0,\end{aligned}\tag{1.1}$$

plus the initial data  $u(0) = u_0$ . With no loss of generality, we assume that  $\nabla \cdot f = 0$ , and that the integrals of  $f$  and  $u_0$  vanish over  $\Omega$ . The last assumption implies that  $u = u(t)$ , whenever it is well-defined for  $t \geq 0$ , also has vanishing integral over  $\Omega$ , giving us the Poincaré inequality

$$|u|_{L^2}^2 \leq c_0(\Omega) |\nabla u|_{L^2}^2.\tag{1.2}$$

For notational convenience, we redefine  $c_0$  to give also the bound

$$|\nabla u|_{L^2}^2 \leq c_0 |\Delta u|_{L^2}^2.\tag{1.3}$$

In order to facilitate comparison with the numerical solutions, in the rest of this section we briefly review the boundedness of solutions of the 3d NSE, both

in  $L^2$  and in  $H^1$  for the two cases (small data, large time and short time for arbitrary data).

Multiplying (1.1) by  $u$  in  $L^2(\Omega)$ , integrating by parts and using the fact that  $(u \cdot \nabla u, u) = 0$ , we find

$$\frac{1}{2} \frac{d}{dt} |u|^2 + \nu |\nabla u|^2 = (f, u). \quad (1.4)$$

35 Here and henceforth, unadorned norm  $|\cdot|$  and inner product  $(\cdot, \cdot)$  are taken to be  $L^2$ . Bounding the rhs by the Cauchy–Schwarz inequality and using the Poincaré inequality, (1.4) becomes

$$\frac{d}{dt} |u|^2 + \frac{\nu}{c_0} |u|^2 \leq \frac{1}{\nu} |f|_{L^\infty(H^{-1})}^2, \quad (1.5)$$

where  $|f|_{L^\infty(H^{-1})} := \sup_{t \geq 0} |f(t)|_{H^{-1}}$ . Hence, we have

$$\frac{d}{dt} \left( |u|^2 \exp \left( \frac{\nu}{c_0} t \right) \right) \leq \exp \left( \frac{\nu}{c_0} t \right) \frac{1}{\nu} |f|_{L^\infty(H^{-1})}^2. \quad (1.6)$$

Integrating from 0 to  $t$  (here we change the dummy variable of integration to  
40  $s$ ), we obtain

$$|u(t)|^2 \exp \left( \frac{\nu}{c_0} t \right) \leq |u(0)|^2 + \frac{c_0}{\nu^2} |f|_{L^\infty(H^{-1})}^2 \left( \exp \left( \frac{\nu}{c_0} t \right) - 1 \right). \quad (1.7)$$

We then find the uniform  $L^2$  bound valid for all  $t \geq 0$ :

$$|u(t)|^2 \leq |u(0)|^2 + (c_0/\nu^2) |f|_{L^\infty(H^{-1})}^2 =: K_0(u_0, f; \nu, \Omega). \quad (1.8)$$

### 1.1. $H^1$ estimate for small data

Now multiplying (1.1) by  $-\Delta u$  in  $L^2(\Omega)$  and integrating by parts, we find

$$\frac{1}{2} \frac{d}{dt} |\nabla u|^2 + \nu |\Delta u|^2 = (u \cdot \nabla u, \Delta u) - (f, \Delta u). \quad (1.9)$$

Bounding the nonlinear term using the Sobolev inequality  $|u|_{L^6} \leq c |\nabla u|_{L^2}$ ,  
45 which is specific to dimension three, we find

$$|(u \cdot \nabla u, \Delta u)| \leq |u|_{L^3} |\nabla u|_{L^6} |\Delta u|_{L^2} \leq \frac{c_1}{2} |u|_{L^3} |\Delta u|_{L^2}^2, \quad (1.10)$$

and the forcing term in the obvious fashion

$$(f, \Delta u) \leq \frac{1}{\nu} |f|^2 + \frac{\nu}{4} |\Delta u|^2. \quad (1.11)$$

We then arrive at

$$\frac{d}{dt}|\nabla u|^2 + (3\nu/2 - c_1|u|_{L^3})|\Delta u|^2 \leq \frac{2}{\nu}|f|^2. \quad (1.12)$$

Assuming that

$$|u_0|_{L^3} \leq \frac{\nu}{4c_1}, \quad (1.13)$$

we find that on some interval of time  $(0, T)$

$$|u|_{L^3} \leq \nu/(2c_1). \quad (1.14)$$

50 We then find that

$$3\nu/2 - c_1|u|_{L^3} \geq \nu. \quad (1.15)$$

Using the Poincaré inequality, (1.12) becomes on this interval

$$\frac{d}{dt}|\nabla u|^2 + \frac{\nu}{c_0}|\nabla u|^2 \leq \frac{2}{\nu}|f|_{L^\infty(L^2)}^2, \quad (1.16)$$

where  $|f|_{L^\infty(L^2)} := \sup_{t \geq 0} |f(t)|_{L^2}$ . We then have

$$\frac{d}{dt} \left( |\nabla u|^2 \exp \left( \frac{\nu}{c_0} t \right) \right) \leq \exp \left( \frac{\nu}{c_0} t \right) \frac{2}{\nu} |f|_{L^\infty(L^2)}^2. \quad (1.17)$$

Integrating from 0 to  $t$  (here we change the dummy variable of integration to  $s$ ), we deduce that

$$|\nabla u(t)|^2 \exp \left( \frac{\nu}{c_0} t \right) \leq |\nabla u(0)|^2 + \frac{2c_0}{\nu^2} |f|_{L^\infty(L^2)}^2 \left( \exp \left( \frac{\nu}{c_0} t \right) - 1 \right). \quad (1.18)$$

55 We obtain

$$|\nabla u(t)|^2 \leq |\nabla u(0)|^2 + (2c_0/\nu^2) |f|_{L^\infty(L^2)}^2 =: K_1(u_0, f; \nu, \Omega). \quad (1.19)$$

Using the Sobolev inequality  $|u|_{L^3}^2 \leq c|u|_{H^{1/2}}^2 \leq c|u||\nabla u|$ , we find that (1.14) and then (1.19) hold for all time provided.

$$K_0 K_1 = (|u_0|^2 + c_0 |f|_{L^\infty(H^{-1})}^2) (|\nabla u_0|^2 + 2c_0 |f|_{L^\infty(L^2)}^2 / \nu^2) \leq c_2(\Omega) \nu^4, \quad (1.20)$$

where  $c_2 = c_2(\Omega) = c/c_1^4$ . It therefore follows that whenever this holds, the 3d NSE has a global solution bounded by (1.8) and (1.19).

60 *1.2.  $H^1$  estimate for short times*

Let us now recall the uniform  $H^1$  estimate for short time and arbitrary data. Multiplying (1.1) by  $-\Delta u$  in  $L^2(\Omega)$  and integrating by parts, we find

$$\frac{1}{2} \frac{d}{dt} |\nabla u|^2 + \nu |\Delta u|^2 = (u \cdot \nabla u, \Delta u) - (f, \Delta u). \quad (1.21)$$

Bounding as before the nonlinear term using the Sobolev and interpolation inequalities, we find

$$\begin{aligned} |(u \cdot \nabla u, \Delta u)| &\leq |u|_{L^6} |\nabla u|_{L^3} |\Delta u|_{L^2} \leq c |\nabla u| |\nabla u|_{H^{1/2}} |\Delta u| \\ &\leq c |\nabla u|^{3/2} |\Delta u|^{3/2} \leq \frac{c_4}{2\nu^3} |\nabla u|^6 + \frac{\nu}{2} |\Delta u|^2, \end{aligned} \quad (1.22)$$

65 and using the Cauchy–Schwarz inequality for the last term, this gives

$$\frac{d}{dt} |\nabla u|^2 \leq \frac{c_4}{\nu^3} |\nabla u|^6 + \frac{1}{\nu} |f|_{L^\infty(L^2)}^2. \quad (1.23)$$

This implies

$$\frac{d}{dt} (|\nabla u|^2 + F) \leq \frac{c_4}{\nu^3} (|\nabla u|^2 + F)^3, \quad (1.24)$$

where  $F := (\nu^2 |f|_{L^\infty(L^2)}^2 / c_4)^{1/3}$ . Writing  $z(t) := |\nabla u(t)|^2 + F$  and integrating, we find

$$\frac{z(t)^2}{z(0)^2} \leq \frac{1}{1 - 2tc_4 z(0)^2 / \nu^3}, \quad (1.25)$$

as long as  $t < \nu^3 / (2c_4 z(0)^2)$ . It is clear from this that our solution will remain  
70 bounded, say,  $z(t)^2 \leq 2z(0)^2$ , for  $0 \leq t \leq \nu^3 / (4c_4 z(0)^2)$ .

In view of results in Sections 1.1 and 1.2, we aim to study the  $H^1$  stability of two different time discretizaion schemes in the following sections; namely, the semi-implicit and the fully implicit Euler scheme.

## 2. Semi-implicit scheme

75 Given a fixed  $k = \Delta t > 0$ , we discretise (1.1) in time using the semi-implicit Euler scheme

$$\frac{u^n - u^{n-1}}{k} + u^{n-1} \cdot \nabla u^n + \nabla p = \nu \Delta u^n + f^n, \quad (2.1)$$

where  $u^n$  and  $f^n$  are approximations such that

$$u^n = u(n\Delta t), \quad f^n = \frac{1}{k} \int_{(n-1)k}^{nk} f(t) dt. \quad (2.2)$$

For the 2d NSE, this scheme was proved in [4] to be globally stable in  $H^1$ . For the 3d NSE, its stability mirrors that (which is known) of the continuous  
80 system, subject to relatively mild timestep restrictions.

We note a few facts that will be useful later on. First, for any  $a$  and  $b \in L^2$ ,

$$2(a - b, a) = |a|^2 - |b|^2 + |a - b|^2. \quad (2.3)$$

Next, for  $b > 0$  and given positive real sequences  $(x_n)$  and  $(r_n)$  satisfying

$$(1 + b)x_n \leq x_{n-1} + r_{n-1}, \quad (2.4)$$

we have

$$x_n \leq (1 + b)^{-n} x_0 + \frac{1 + b}{b} \max_j r_j. \quad (2.5)$$

The  $L^2$  bound works out essentially as in the continuous case: multiplying  
85 (2.1) by  $2ku^n$ , using (2.3) and noting that  $(u^{n-1} \cdot \nabla u^n, u^n) = 0$ , we find

$$|u^n|^2 + |u^n - u^{n-1}|^2 + 2\nu k |\nabla u^n|^2 = |u^{n-1}|^2 + 2k(f^n, u^n). \quad (2.6)$$

Bounding the forcing term using Cauchy–Schwarz, we obtain

$$\begin{aligned} |u^n|^2 + 2\nu k |\nabla u^n|^2 &\leq |u^{n-1}|^2 + 2k(f^n, u^n) \\ &\leq |u^{n-1}|^2 + k|f^n|_{H^{-1}}^2/\nu + k\nu |\nabla u^n|^2. \end{aligned} \quad (2.7)$$

Using the Poincaré inequality, we deduce that

$$(1 + \nu k/c_0)|u^n|^2 \leq |u^{n-1}|^2 + k|f^n|_{H^{-1}}^2/\nu. \quad (2.8)$$

Integrating this using (2.5), we find for all  $n \in \{1, 2, \dots\}$ ,

$$\begin{aligned} |u^n|^2 &\leq |u^0|^2 + \frac{c_0 + \nu k}{\nu^2} |f|_{L^\infty(H^{-1})}^2 \\ &= K_0(u_0, f; \nu, \Omega) + (k/\nu) |f|_{L^\infty(H^{-1})}^2, \end{aligned} \quad (2.9)$$

where  $K_0$ , and  $K_1$  below, are as in the continuous case. We note that this bound  
90 (the rhs of (2.9)) tends to  $K_0$  as  $k \rightarrow 0$ .

### 2.1. $H^1$ estimate for small data

**Theorem 1.** *For small data, let the initial data  $u_0 \in H^1$ , the forcing  $f$  and the timestep  $k$  satisfy*

$$(K_0 + k|f|_{L^\infty(H^{-1})}^2/\nu)(K_1 + 2k|f|_{L^\infty(L^2)}^2/\nu) \leq c_2(\Omega)\nu^4, \quad (2.10)$$

where  $K_0(u_0, f)$  and  $K_1(u_0, f)$  are as in the continuous case, (1.8) and (1.19).

<sup>95</sup> Then  $u^n$  is bounded in  $H^1$  as follows,

$$|\nabla u^n|^2 \leq K_1 + (2k/\nu)|f|_{L^\infty(L^2)}^2 \quad \text{for all } n \geq 0. \quad (2.11)$$

*Proof.* We now turn to stability in  $H^1$  for small solutions. Multiplying (2.1) by  $-2k\Delta u^n$  and using (2.3), we find

$$\begin{aligned} |\nabla u^n|^2 + |\nabla(u^n - u^{n-1})| + 2\nu k|\Delta u^n|^2 \\ = |\nabla u^{n-1}|^2 - 2k(f^n, \Delta u^n) + 2k(u^{n-1} \cdot \nabla u^n, \Delta u^n). \end{aligned} \quad (2.12)$$

Bounding the nonlinear term using the Sobolev inequality,

$$|(u^{n-1} \cdot \nabla u^n, \Delta u^n)| \leq |u^{n-1}|_{L^3} |\nabla u^n|_{L^6} |\Delta u^n|_{L^2} \leq c_1 |u^{n-1}|_{L^3} |\Delta u^n|_{L^2}^2, \quad (2.13)$$

and using the Cauchy–Schwarz inequality for the forcing, (2.12) implies

$$|\nabla u^n|^2 + (3\nu/2 - c_1 |u^{n-1}|_{L^3})k|\Delta u^n|^2 \leq |\nabla u^{n-1}|^2 + (2k/\nu)|f^n|^2. \quad (2.14)$$

<sup>100</sup> As in (1.14) of Section 1.1, if we now assume that

$$|u^{n-1}|_{L^3} \leq \nu/(2c_1), \quad (2.15)$$

we deduce from (2.14) that

$$(1 + \nu k/c_0)|\nabla u^n|^2 \leq |\nabla u^{n-1}|^2 + 2k|f^n|^2/\nu. \quad (2.16)$$

As long as (2.15) holds, we can integrate this using (2.5) to get the bound

$$\begin{aligned} |\nabla u^n|^2 &\leq |\nabla u^0|^2 + \frac{2(c_0 + \nu k)}{\nu^2} |f|_{L^\infty(L^2)}^2 \\ &\leq K_1(u_0, f; \nu, \Omega) + (2k/\nu)|f|_{L^\infty(L^2)}^2, \end{aligned} \quad (2.17)$$



which proves (2.11). As in the continuous case, we now use the Sobolev inequality and interpolation inequalities to bound

$$|u^{n-1}|_{L^3}^2 \leq c |u^{n-1}|_{H^{1/2}}^2 \leq c |u^{n-1}| |\nabla u^{n-1}|. \quad (2.18)$$

105 The timestep restriction (2.10) then becomes a sufficient condition for (2.15). More explicitly, since (2.10) holds at  $n = 0$ , (2.9) and (2.11) imply that it will hold at  $n = 1$  and, by induction, for all  $n \in \{2, \dots\}$ , i.e. the solution of the scheme (2.1) is bounded uniformly in (discrete) time subject to (2.10). Comparing to (1.20), we note that this condition also depends on the timestep  
110  $k$  in addition to  $u_0$  and  $f$ . This timestep restriction is however relatively mild compared to that for the fully implicit scheme in §3 below.  $\square$

## 2.2. $H^1$ estimate for short times

**Theorem 2.** *For short times, assuming the timestep restriction (2.26), we have*

$$|\nabla u^n|^2 \leq 2 |\nabla u^0|^2 + F, \quad \text{where } F := (\nu^2 |f|_{L^\infty(L^2)}^2 / c_4)^{1/3}, \quad (2.19)$$

for all  $n$  such that  $nk = t_n \leq \nu^3 / (8c_4(|\nabla u^0|^2 + F)^2)$ .

115 *Proof.* For short-time  $H^1$  stability, we bound the nonlinear term in (2.12) as in (1.22),

$$\begin{aligned} |(u^{n-1} \cdot \nabla u^n, \Delta u^n)| &\leq |u^{n-1}|_{L^6} |\nabla u^n|_{L^3} |\Delta u^n|_{L^2} \\ &\leq c |\nabla u^{n-1}| |\nabla u^n|_{H^{1/2}} |\Delta u^n| \\ &\leq c |\nabla u^{n-1}| |\nabla u^n|^{1/2} |\Delta u^n|^{3/2} \\ &\leq \frac{c_4}{2\nu^3} |\nabla u^{n-1}|^4 |\nabla u^n|^2 + \frac{\nu}{2} |\Delta u^n|^2. \end{aligned} \quad (2.20)$$

Then, (2.12) becomes

$$\begin{aligned} &|\nabla u^n|^2 + 2\nu k |\Delta u^n|^2 \\ &\leq |\nabla u^{n-1}|^2 - 2k(f^n, \Delta u^n) + k\nu |\Delta u^n|^2 + \frac{kc_4}{\nu^3} |\nabla u^{n-1}|^4 |\nabla u^n|^2, \end{aligned} \quad (2.21)$$

this implies

$$|\nabla u^n|^2 \leq |\nabla u^{n-1}|^2 + \frac{c_4 k}{\nu^3} |\nabla u^{n-1}|^4 |\nabla u^n|^2 + \frac{k}{\nu} |f|_{L^\infty(L^2)}^2. \quad (2.22)$$

We can rewrite (2.22) as

$$\begin{aligned} & \left(1 - \frac{c_4 k}{\nu^3} |\nabla u^{n-1}|^4\right) |\nabla u^n|^2 \\ & \leq \left(1 - \frac{c_4 k}{\nu^3} |\nabla u^{n-1}|^4\right) |\nabla u^{n-1}|^2 + \frac{c_4 k}{\nu^3} |\nabla u^{n-1}|^6 + \frac{k}{\nu} |f|_{L^\infty(L^2)}^2. \end{aligned} \quad (2.23)$$

120 Since we are interested in short times, we assume that  $|\nabla u^{n-1}|^2 \leq 2|\nabla u^0|^2$  for all relevant  $n$  and demand that  $k$  satisfy

$$k \leq \frac{\nu^3}{8c_4 |\nabla u^0|^4}. \quad (2.24)$$

This implies that the brackets in (2.23) are  $\geq \frac{1}{2}$ , that is

$$\frac{1}{2} \leq 1 - \frac{4c_4 k}{\nu^3} |\nabla u^0|^4 \leq 1 - \frac{c_4 k}{\nu^3} |\nabla u^{n-1}|^4; \quad (2.25)$$

For later use we add the extra  $F$ , which make a stronger condition than (2.24)

$$k \leq \frac{\nu^3}{2c_4 (2|\nabla u^0|^2 + F)^2}. \quad (2.26)$$

With this assumption, (2.23) implies

$$\frac{|\nabla u^n|^2 - |\nabla u^{n-1}|^2}{k} \leq \frac{2c_4}{\nu^3} |\nabla u^{n-1}|^6 + \frac{2}{\nu} |f|_{L^\infty(L^2)}^2. \quad (2.27)$$

125 Unlike its continuous-time analogue (1.23), this difference inequality implies  $|\nabla u^n| < \infty$  for all  $n$ , although for sufficiently large time  $nk$ , it (i.e. the bound) grows without bound as  $k \rightarrow 0$ . This is a well-known pitfall in discretising differential equations in time. To obtain a finite-time bound on  $|\nabla u^n|$ , we proceed in analogy with (1.24) and define

$$z_n := |\nabla u^n|^2 + F. \quad (2.28)$$

130 We then get from (2.27)

$$\frac{z_n - z_{n-1}}{k} \leq \frac{2c_4}{\nu^3} z_{n-1}^3 =: g(z_{n-1}). \quad (2.29)$$

Observe that  $g(\zeta) > g(\hat{\zeta})$  whenever  $\zeta > \hat{\zeta}$ , that is,  $g \geq 0$  is a strictly monotone increasing function.

Now let  $\zeta_n$  be the positive solution of the difference equation,

$$\frac{\zeta_n - \zeta_{n-1}}{k} = g(\zeta_{n-1}), \quad (2.30)$$

and observe that  $\zeta_n \geq 0$  if  $\zeta_{n-1} \geq 0$ . Denoting  $t_n := nk$ , we claim that

$$\zeta_n \leq \zeta(t_n) \quad (2.31)$$

135 where  $\zeta(\cdot)$  is the solution of the differential equation

$$\frac{d\zeta}{dt} = g(\zeta) \quad \text{with } \zeta(t_{n-1}) = \zeta_{n-1}. \quad (2.32)$$

To show this, we first note that  $\zeta(t)$  is non-decreasing since  $g \geq 0$ . Then

$$\zeta(t_n) - \zeta(t_{n-1}) = \int_{t_{n-1}}^{t_n} g(\zeta(t)) dt \geq \int_{t_{n-1}}^{t_n} g(\zeta(t_{n-1})) dt = kg(\zeta_{n-1}). \quad (2.33)$$

Thanks to (2.30), we obtain that

$$\zeta(t_n) - \zeta(t_{n-1}) = \zeta_n - \zeta_{n-1}, \quad (2.34)$$

and this prove our claim. By induction, taking  $\zeta(0) = \zeta_0 > 0$  instead of the initial data in (2.30), we then have  $\zeta_n \leq \zeta(t_n)$  for all  $n \in \{1, 2, \dots\}$ . Comparing

140 with the continuous case (1.24)–(1.25), we conclude that  $\zeta_n \leq \zeta(t_n) \leq 2\zeta(0) = 2\zeta_0$  for  $nk = t_n \leq \nu^3/(8c_4\zeta_0^2)$ .

Taking  $\zeta_0 = z_0$ , clearly  $z_n \leq \zeta_n$  for all  $n \geq 0$ . We therefore have

$$\begin{aligned} z_n &= |\nabla u^n|^2 + F \leq \zeta_n \leq 2\zeta_0 = 2|\nabla u^0|^2 + 2F \\ \implies |\nabla u^n|^2 &\leq 2|\nabla u^0|^2 + F, \end{aligned} \quad (2.35)$$

for all  $n$  such that  $nk = t_n \leq \nu^3/(8c_4(|\nabla u^0|^2 + F)^2)$ , which is half as long as the bound in the continuous case.  $\square$

### 145 3. Fully implicit scheme

We now consider the fully implicit Euler scheme

$$\frac{u^n - u^{n-1}}{k} + u^n \cdot \nabla u^n + \nabla p = \nu \Delta u^n + f^n, \quad (3.1)$$

with  $\nabla \cdot u^n = 0$  for all  $n$  and  $u^0 = u_0$ . In two space dimensions, uniform boundedness in  $H^1$  for this scheme was proved in [8].

The  $L^2$  bound obtains as before: multiplying (3.1)<sub>1</sub> by  $2ku^n$  and using (2.3),

$$|u^n|^2 + |u^n - u^{n-1}|^2 + 2\nu k |\nabla u^n|^2 = |u^{n-1}|^2 + 2k(f^n, u^n). \quad (3.2)$$

150 Bounding the forcing term in the obvious manner and using Poincaré, this implies

$$(1 + \nu k/c_0)|u^n|^2 \leq |u^{n-1}|^2 + k|f^n|_{H^{-1}}^2/\nu. \quad (3.3)$$

Integrating this using (2.5), we find for all  $n \in \{1, 2, \dots\}$ ,

$$\begin{aligned} |u^n|^2 &\leq |u^0|^2 + \frac{c_0 + \nu k}{\nu^2} |f|_{L^\infty(H^{-1})}^2 \\ &= K_0(u_0, f; \nu, \Omega) + (k/\nu) |f|_{L^\infty(H^{-1})}^2. \end{aligned} \quad (3.4)$$

As before, this bound tends to  $K_0$  as  $k \rightarrow 0$ . For later use, we define

$$\tilde{K}_0(u_0, f; \nu, \Omega) := |u_0|^2 + \frac{2c_0}{\nu^2} |f|_{L^\infty(H^{-1})}^2. \quad (3.5)$$

The central ingredient for our main results of the  $H^1$  stability is the following

155 local-in-time estimate:

**Lemma 1.** *We assume the  $L^2$  uniform bound (3.4) and that  $u^{n-1} \in H^1$ . Assuming further the timestep restrictions*

$$K^{(n-1)} \leq \frac{1}{2} \left( \frac{\nu^3}{3c_4 k} \right)^{1/2}, \quad (3.6)$$

$$\left( 1 + \frac{c_5}{\nu^4} \tilde{K}_0 K^{(n-1)} \right) K^{(n-1)} + |f|_{L^\infty(H^{-1})}^2/\nu^2 \leq \left( \frac{\nu^3}{12c_4 k} \right)^{1/2}, \quad (3.7)$$

where  $K^{(n-1)} := |\nabla u^{n-1}|^2 + (10c_0/\nu) |f|_{L^\infty(L^2)}^2$ , then the solution  $u^n$  of (3.1) is bounded as  $|\nabla u^n|^2 \leq y_1$  where  $y_1$  is the smallest positive root of the cubic equation (3.10).

*Proof.* Multiplying (3.1) by  $-2k\Delta u^n$ , we have

$$\begin{aligned} |\nabla u^n|^2 + |\nabla(u^n - u^{n-1})|^2 + 2\nu k |\Delta u^n|^2 \\ = |\nabla u^{n-1}|^2 + 2k(u^n \cdot \nabla u^n, \Delta u^n) - 2k(f^n, \Delta u^n). \end{aligned} \quad (3.8)$$

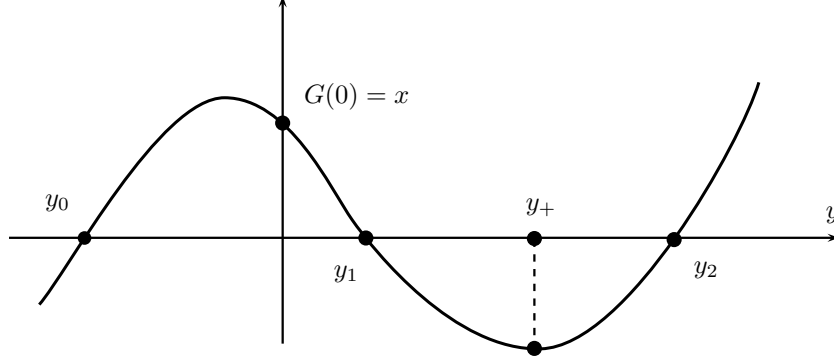


Figure 1: The graph of  $G(y)$  in (3.11):  $y_+$  is a local minimum.

160 Bounding the nonlinear term as we did in (1.22),

$$2k|(u^n \cdot \nabla u^n, \Delta u^n)| \leq \frac{c_4 k}{\nu^3} |\nabla u^n|^6 + \nu k |\Delta u^n|^2, \quad (3.9)$$

we find

$$\begin{aligned} 0 &\leq \frac{c_4 k}{\nu^3} |\nabla u^n|^6 - |\nabla u^n|^2 - \frac{\nu k}{2} |\Delta u^n|^2 + |\nabla u^{n-1}|^2 + \frac{2k}{\nu} |f^n|^2 \\ \Rightarrow 0 &\leq \frac{c_4 k}{\nu^3} |\nabla u^n|^6 - \left(1 + \frac{\nu k}{2c_0}\right) |\nabla u^n|^2 + |\nabla u^{n-1}|^2 + \frac{2k}{\nu} |f|_{L^\infty(L^2)}^2. \end{aligned} \quad (3.10)$$

Let  $y := |\nabla u^n|^2$ ,  $x := |\nabla u^{n-1}|^2 + 2k|f|_{L^\infty(L^2)}^2/\nu$  and

$$G(y; x) := (c_4 k/\nu^3)y^3 - (1 + \nu k/(2c_0))y + x. \quad (3.11)$$

We write  $G(y)$  instead of  $G(y; x)$  when there is no risk of confusion. We are of course interested in the solution set of  $G(y) \geq 0$ .

Under the assumption (3.14) below on the timestep  $k$ , the graph of the cubic  
165  $G$  is (qualitatively) as shown in Figure 1. We note in particular that  $G(y) = 0$  has a negative root  $y_0$  and two positive roots  $y_1$  and  $y_2$ . To verify this, we note the following. First,  $G(y) \rightarrow \pm\infty$  as  $y \rightarrow \pm\infty$ . Next,  $G(y)$  has two local extrema,

$$y_{\pm} = \pm \left( \frac{\nu^3}{3c_4 k} \left[ 1 + \frac{\nu k}{2c_0} \right] \right)^{1/2}, \quad (3.12)$$

with  $y_- < 0$  being a local maximum and  $y_+ > 0$  a local minimum, as verified  
170 by computing  $G''(y_{\pm})$ . Since  $G(0) = x > 0$  (the problem is trivial if  $x = 0$ ), we

have  $G(y_-) > 0$ . Finally, computing

$$G(y_+) = -\frac{2}{3}\left(1 + \frac{\nu k}{2c_0}\right)\left(\frac{\nu^3}{3c_4 k}\left[1 + \frac{\nu k}{2c_0}\right]\right)^{1/2} + x, \quad (3.13)$$

we conclude that  $G(y_+) < 0$  if (this is essentially a restriction on  $k$ )

$$|\nabla u^{n-1}|^2 + \frac{2k}{\nu}|f|_{L^\infty(L^2)}^2 < \frac{2}{3}\left(\frac{\nu^3}{3c_4 k}\right)^{1/2}. \quad (3.14)$$

This implies the existence of the two positive roots  $y_1$  and  $y_2$  with  $y_1 < y_+ < y_2$ .

Now (3.10) implies that  $|\nabla u^n|^2 = y$  lies in the disjoint set  $[0, y_1] \cup [y_2, \infty)$ .

175 However,  $y_2 > y_+ \sim k^{-1/2}$ , which is absurd for small  $k$ . To prove that  $y \notin [y_2, \infty)$ , we multiply (3.1)<sub>1</sub> by  $2k(u^n - u^{n-1})$  in  $L^2$  to get

$$\begin{aligned} 2|u^n - u^{n-1}|^2 + \nu k|\nabla u^n|^2 - \nu k|\nabla u^{n-1}|^2 + \nu k|\nabla(u^n - u^{n-1})|^2 \\ = -2k(u^n \cdot \nabla u^n, u^n - u^{n-1}) + 2k(f^n, u^n - u^{n-1}) =: I_1 + I_2. \end{aligned} \quad (3.15)$$

Bounding the rhs as

$$\begin{aligned} |I_2| &\leq \frac{k}{\nu}|f^n|_{H^{-1}}^2 + \nu k|\nabla(u^n - u^{n-1})|^2 \\ |I_1| &= 2k|(u^n \cdot \nabla u^n, u^{n-1})| \leq 2k|u^n|_{L^3}|\nabla u^n|_{L^2}|u^{n-1}|_{L^6} \\ &\leq ck|u^n|_{H^{1/2}}|\nabla u^n||\nabla u^{n-1}| \leq ck|u^n|^{1/2}|\nabla u^n|^{3/2}|\nabla u^{n-1}| \\ &\leq \frac{\nu k}{2}|\nabla u^n|^2 + \frac{ck}{\nu^3}|u^n|^2|\nabla u^{n-1}|^4, \end{aligned}$$

and dropping the  $2|u^n - u^{n-1}|^2$  on the lhs in (3.15), we arrive at

$$|\nabla u^n|^2 \leq \left(2 + \frac{2c_5}{\nu^4}|u^n|^2|\nabla u^{n-1}|^2\right)|\nabla u^{n-1}|^2 + \frac{2}{\nu^2}|f|_{L^\infty(H^{-1})}^2. \quad (3.16)$$

If we now assume that (effectively a timestep restriction)

$$\left(2 + \frac{2c_5}{\nu^4}|u^n|^2|\nabla u^{n-1}|^2\right)|\nabla u^{n-1}|^2 + \frac{2}{\nu^2}|f|_{L^\infty(H^{-1})}^2 \leq \left(\frac{\nu^3}{3c_4 k}\right)^{1/2}, \quad (3.17)$$

noting that the rhs  $< y_+ < y_2$ , we can conclude that  $|\nabla u^n|^2 < y_2$  and therefore

180  $|\nabla u^n|^2 \in [0, y_1]$ . This gives us the local  $H^1$  integrability of the scheme (3.1): if  $k$  (is small enough that it) satisfies (3.14) and (3.17), the one-step solution of (3.1) is bounded in  $H^1$ .  $\square$

The crucial point which is not immediately obvious is that  $y_1 = |\nabla u^{n-1}|^2 + O(k)$  for small  $k$ . By estimating the  $O(k)$  more carefully, we obtain our main results.

### 3.1. $H^1$ estimate for small data

**Theorem 3.** *For small data, let  $u_0$  and  $f$  be such that*

$$|\nabla u_0|^2 + \frac{2c_0}{\nu^2} |f|_{L^\infty(L^2)}^2 \leq \frac{\nu^2}{2\sqrt{c_0 c_4}}, \quad (3.18)$$

and let the timestep  $k$  satisfy (3.24)–(3.26) below. Then  $u^n$  is bounded as

$$|\nabla u^n|^2 \leq \tilde{K}_1(u_0, f; \nu, \Omega) := |\nabla u_0|^2 + \frac{10c_0}{\nu^2} |f|_{L^\infty(L^2)}^2, \quad (3.19)$$

for all  $n \in \{0, 1, \dots\}$ .

*Proof.* For small data, we first assume the hypotheses (local in  $n$ ) of Lemma 1. We then derive a more useful explicit bound for  $|\nabla u^{n-1}|^2$ . We claim that with the assumption (3.18),  $|\nabla u^n|^2 \leq y_1$  implies

$$\left(1 + \frac{\nu k}{4c_0}\right) |\nabla u^n|^2 \leq |\nabla u^{n-1}|^2 + \frac{2k}{\nu} |f|_{L^\infty(L^2)}^2, \quad (3.20)$$

where  $y_1$  is the smallest positive root of the cubic equation (3.10). To prove this, we set  $y_* := (|\nabla u^{n-1}|^2 + 2k|f|_{L^\infty(L^2)}^2/\nu)/(1 + \nu k/(4c_0))$  and compute

$$G(y_*) = y_* \left(1 + \frac{\nu k}{4c_0}\right)^{-2} \left\{ -\frac{\nu k}{4c_0} \left(1 + \frac{\nu k}{4c_0}\right)^2 + \frac{c_4 k}{\nu^3} x^2 \right\}, \quad (3.21)$$

where  $G(y)$  is as in (3.11) such that

$$G(y) = (c_4 k/\nu^3) y^3 - (1 + \nu k/(2c_0)) y + x,$$

and  $x := |\nabla u^{n-1}|^2 + 2k|f|_{L^\infty(L^2)}^2/\nu$ . Now  $G(y_*) \leq 0$  implies that  $y_* \geq y_1$ , and the former is true if

$$-\frac{\nu k}{4c_0} \left(1 + \frac{\nu k}{4c_0}\right)^2 + \frac{c_4 k}{\nu^3} x^2 \leq 0,$$

and this implies

$$|\nabla u^{n-1}|^2 + \frac{2k}{\nu} |f|_{L^\infty(L^2)}^2 = x \leq \frac{\nu^2}{2\sqrt{c_0 c_4}}. \quad (3.22)$$

Hence,  $|\nabla u^n|^2 \leq y_1 \leq y_*$  implies (3.20). Then, the claim is proved.

To obtain the uniform bound, we sum (3.20) using (2.5) to find

$$|\nabla u^n|^2 \leq \left(1 + \frac{\nu k}{4c_0}\right)^{-n} |\nabla u^0|^2 + \frac{8c_0}{\nu^2} |f|_{L^\infty(L^2)}^2 + \frac{2k}{\nu} |f|_{L^\infty(L^2)}^2. \quad (3.23)$$

Assuming that

$$k \leq c_0/\nu, \quad (3.24)$$

we can absorb the last term into the penultimate one to obtain (3.19). Consolidating our assumptions, the smallness condition (3.22) is now implied by (3.18), while the timestep restrictions (3.14) and (3.17) can both be satisfied by taking  $k$  sufficiently small to satisfy

$$\tilde{K}_1 \leq \frac{1}{2} \left( \frac{\nu^3}{3c_4 k} \right)^{1/2}, \quad (3.25)$$

$$\left(1 + \frac{c_5}{\nu^4} \tilde{K}_0 \tilde{K}_1\right) \tilde{K}_1 + |f|_{L^\infty(H^{-1})}^2 / \nu^2 \leq \left( \frac{\nu^3}{12c_4 k} \right)^{1/2}. \quad (3.26)$$

This proves the theorem.  $\square$

200 We note that, up to the constant depending only on the domain  $\Omega$ , the bound (3.19) is the same as that in the continuous case (1.19). In addition, considering  $|f|_{H^{-1}}^2 \leq c_0^* |f|_{L^2}^2$  and using Poincaré inequality, we estimate  $K_0$  in (1.20) such that

$$K_0 := |u_0|^2 + c_0 |f|_{L^\infty(H^{-1})}^2 \leq c_0 |\nabla u_0|^2 + c_0 c_0^* |f|_{L^\infty(L^2)}^2. \quad (3.27)$$

Assuming  $k \leq (c_0^* \nu)/2$ , we deduce from (3.22) that

$$c_0 \left( |\nabla u_0|^2 + c_0^* |f|_{L^\infty(L^2)}^2 \right) \leq \frac{\nu^2 \sqrt{c_0}}{2\sqrt{c_4}}. \quad (3.28)$$

205 From (3.18) and (3.28), we find, up to the constant depending only on the domain  $\Omega$ , the same smallness condition as in (1.20).

### 3.2. $H^1$ estimate for short times

**Theorem 4.** *For short times, let the timestep  $k$  satisfy (3.40), (3.41) and (3.42) below. Then there exists a  $t_f^* = \nu^3/(8c_4\zeta_0^2)$ , as long as  $0 \leq nk \leq t_f^*$  we have*

$$|\nabla u^n|^2 \leq 2 |\nabla u_0|^2 + (\nu^2 |f|_{L^\infty(L^2)}^2 / c_4)^{1/3}. \quad (3.29)$$



210 *Proof.* We first assume the hypotheses (local in  $n$ ) of Lemma 1. Since  $y_1$  is the root of a cubic  $G(y_1) = 0$ , the bound  $|\nabla u^n|^2 \leq y_1$  is not very convenient, so we compute a more useful bound. Recalling that  $x > 0$ , we consider for some  $a > 0$

$$\begin{aligned} G((1+ak)x; x) &= xk \left[ \frac{c_4}{\nu^3} (1+ak)^3 x^2 - \left( \frac{\nu}{2c_0} + a \right) - \frac{a\nu k}{2c_0} \right] \\ &< xk \left[ \frac{c_4}{\nu^3} (1+ak)^3 x^2 - a \right]. \end{aligned} \quad (3.30)$$

Setting  $a = 2c_4 x^2 / \nu^3$ , this implies  $G((1+ak)x) < 0$  if

$$1 + ak \leq 2^{1/3} \quad \Leftrightarrow \quad (|\nabla u^{n-1}|^2 + 2k|f|_{L^\infty(L^2)}^2 / \nu)^2 2c_4 k / \nu^3 \leq 2^{1/3} - 1. \quad (3.31)$$

Assuming this, Lemma 1 then gives us the explicit one-step estimate

$$\begin{aligned} |\nabla u^n|^2 &\leq y_1 \leq (1+ak)x \\ &= |\nabla u^{n-1}|^2 + \frac{2k}{\nu} |f|_{L^\infty(L^2)}^2 + \frac{2c_4 k}{\nu^3} (|\nabla u^{n-1}|^2 + (2k/\nu) |f|_{L^\infty(L^2)}^2)^3, \end{aligned} \quad (3.32)$$

215 which we can rewrite as

$$\frac{|\nabla u^n|^2 - |\nabla u^{n-1}|^2}{k} \leq \frac{2c_4}{\nu^3} \left[ \left( |\nabla u^{n-1}|^2 + \frac{2k}{\nu} |f|_{L^\infty(L^2)}^2 \right)^3 + \frac{\nu^2}{c_4} |f|_{L^\infty(L^2)}^2 \right]. \quad (3.33)$$

To obtain a finite-time bound on  $|\nabla u^n|$ , we proceed in analogy with (1.24) and define

$$z_n := |\nabla u^n|^2 + F \quad \text{where } F^3 = \frac{2\nu^2}{c_4} |f|_{L^\infty(L^2)}^2. \quad (3.34)$$

By expanding both sides, we have

$$\left( |\nabla u^{n-1}|^2 + \frac{2k}{\nu} |f|_{L^\infty(L^2)}^2 \right)^3 + \frac{\nu^2}{c_4} |f|_{L^\infty(L^2)}^2 \leq z_{n-1}^3, \quad (3.35)$$

subject to the timestep restriction

$$k \leq \frac{\nu^{5/3}}{2c_4^{1/3} |f|_{L^\infty(L^2)}^{4/3}} \quad \Rightarrow \quad \begin{cases} 4^{1/3} c_4^{1/3} |f|_{L^\infty(L^2)}^{4/3} k \leq \nu^{5/3}, \\ 4^{2/3} c_4^{2/3} |f|_{L^\infty(L^2)}^{8/3} k^2 \leq \nu^{10/3}, \\ 8c_4 |f|_{L^\infty(L^2)}^4 k^3 \leq \nu^5. \end{cases} \quad (3.36)$$

220 The time step restriction above can be found by direct computations. Indeed,

expanding (3.35), we have

$$\begin{aligned}
& |\nabla u^{n-1}|^6 + 3|\nabla u^{n-1}|^4 \left( \frac{2k}{\nu} |f|_{L^\infty(L^2)}^2 \right) + 3|\nabla u^{n-1}|^2 \left( \frac{2k}{\nu} |f|_{L^\infty(L^2)}^2 \right)^2 \\
& \quad + \left( \frac{2k}{\nu} |f|_{L^\infty(L^2)}^2 \right)^3 + \frac{\nu^2}{c_4} |f|_{L^\infty(L^2)}^2 \\
& \leq |\nabla u^{n-1}|^6 + 3|\nabla u^{n-1}|^4 F + 3|\nabla u^{n-1}|^2 F^2 + F^3 \\
& = |\nabla u^{n-1}|^6 + 3|\nabla u^{n-1}|^4 \left( \frac{2\nu^2}{c_4} |f|_{L^\infty(L^2)}^2 \right)^{1/3} \\
& \quad + 3|\nabla u^{n-1}|^2 \left( \frac{2\nu^2}{c_4} |f|_{L^\infty(L^2)}^2 \right)^{2/3} + \frac{2\nu^2}{c_4} |f|_{L^\infty(L^2)}^2.
\end{aligned} \tag{3.37}$$

Hence, assuming the three conditions in the brace of (3.36), the inequality (3.37)

holds term by term. Then (3.32) implies

$$\frac{z_n - z_{n-1}}{k} \leq \frac{2c_4}{\nu^3} z_{n-1}^3 =: g(z_{n-1}). \tag{3.38}$$

Arguing as we did in the semi-implicit case [cf. (2.30)–(2.33)], we conclude that

225  $z_n \leq 2z_0$  for all  $n \geq 0$  such that

$$nk = t_n \leq \nu^3 / (8c_4 \zeta_0^2) =: t_f^*. \tag{3.39}$$

This proves the theorem subject to the timestep restrictions, which we collect here. First, (3.6) and (3.7) are implied by

$$\tilde{K} \leq \frac{1}{2} \left( \frac{\nu^3}{3c_4 k} \right)^{1/2}, \tag{3.40}$$

$$\left( 1 + \frac{c_5}{\nu^4} \tilde{K}_0 \tilde{K} \right) \tilde{K} + |f|_{L^\infty(H^{-1})}^2 / \nu^2 \leq \left( \frac{\nu^3}{12c_4 k} \right)^{1/2}, \tag{3.41}$$

where unlike in Lemma 1, here

$$\tilde{K} := 2|\nabla u^0|^2 + 2(\nu^2 |f|_{L^\infty(L^2)}^2 / c_4)^{1/3} + (10c_0 / \nu) |f|_{L^\infty(L^2)}^2.$$

Next, (3.36) is good as it stands. Finally, using (3.36) to handle the  $k$  inside the bracket, (3.31) is implied by

$$\left( 2|\nabla u^0|^2 + \frac{(1 + 2^{1/3})\nu^{2/3} |f|_{L^\infty(L^2)}^{2/3}}{c_4^{1/3}} \right)^2 \leq \frac{(2^{1/3} - 1)\nu^3}{2c_4 k}. \tag{3.42}$$

This proves the short-time case.

□

230 The time bound  $t_f^*$  is essentially that in the continuous case (smaller by a factor of  $\frac{1}{2}$  which can be improved to  $1 - \varepsilon$  with some work and more restriction on  $k$ ).

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